not taken into account, but this is quite admissible for pores up to several tens of microns in diameter because of the substantial difference between the coefficients of thermal activity of the gaseous and solid phases of the material [8].

The analysis has been limited to the special case of pore collapse in a plastically deforming material. The results obtained give a qualitatively correct description of the functional trends of the temperature distribution over the thickness of the layer. The quantitative results correspond to the phenomenology of the process, except for a spherical layer in the vicinity of the pore, where the effects of viscous flow play an important role.

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CALCULATION OF AN INHOMOGENEOUS ELASTIC HALF SPACE AND A PLATE PLACED ON IT

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In many problems solved within the framework of the model of the linear theory of elasticity of isotropic media, it is necessary to take into account the variation of the properties of the medium as functions of the coordinates, particularly the coordinate z, the depth of a half space. Such problems arise in geophysics, seismology, and structural mechanics. Since the equations of motion of an elastic medium are interrelated, an effective analytic solution of boundary-value problems for any inhomogeneity is very difficult to obtain. However, as is shown in [1], the Lamé vector equation of motion permits separation into independent equations for an infinite set of inhomogeneous media. Assuming weak inhomogeneity of these media does not detract in practice from the generality of the results and, at the same time, permits effective solution of boundary-value problems by approximate methods. Real media can be approximated with sufficient accuracy of the results by media which belong to the above-mentioned set [2]. In the present study we obtain a solution for a problem in the vibrations of an infinite classical plate on an inhomogeneous half space when a vibrating load moves at constant velocity over the plate. We consider in more detail special cases of the problem which are obtained from the previous solution by a passage to the limit with respect to several parameters. When the medium becomes homogeneous, the functional relationships found become the results known for homogeneous media.

1. We consider an isotropic elastic half space in a Cartesian coordinate system with the positive direction of the axis OZ pointing downward. The equation of motion of the medium is taken in the form [1]

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$$\left(\nabla^2 + H_n(z) - \frac{\partial^2}{v_n^2(z)\,\partial t^2}\right)\Psi_n = 0 \quad (n = 1, 2, 3), \tag{1.1}$$

$$H_n(z) = -\frac{1}{2}\left(p_3' + \frac{1}{2}p_3^2\right) + \frac{(-1)^{n-1}}{\gamma^{2(2-n)}(z)}\left(p_n g_n(z) + g_n'(z)(\gamma^2(z) - 1)\right) \; (n = 1, 2),$$

$$H_3(z) = -\frac{1}{2}\left(p_1' + \frac{1}{2}p_1^2\right), \; p_1 = \mu'\mu^{-1}, \; p_2 = \lambda'\mu^{-1}, \; p_3 = \rho'\rho^{-1},$$

$$g_n(z) = (-1)^n\left(2p_1 - p_3\gamma^{2(2-n)}(z)\right)(\gamma^2(z) - 1)^{-1},$$

$$v_n^2(z) = \frac{\mu}{\rho}\gamma^{2(2-n)}(z) \quad (n = 1, 2),$$

$$\gamma^2(z) = (\lambda + 2\mu)(\mu)^{-1},$$

where λ, μ, ρ are the Lamé parameters and the density of the medium, respectively; a prime indicates differentiation with respect to z; ∇^2 is the Laplace operator; the $v_n(z)$ are the velocities of the elastic deformation waves. The vector displacement $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$ is related to the potentials Ψ_n by the equations

$$f_1\mathbf{u}_1 = \nabla\left(f_1\sqrt{\frac{\rho(0)}{\rho(z)}}\Psi_1\right), \; f_2\mathbf{u}_2 = \nabla\times\nabla\times\left(\mathbf{i}_z f_2\sqrt{\frac{\rho(0)}{\rho(z)}}\Psi_2\right),$$

$$\mathbf{u}_3 = \nabla\times\left(\mathbf{i}_z\sqrt{\frac{\mu(0)}{\mu(z)}}\Psi_3\right),$$

where ∇ is the Hamilton operator; $\mathbf{i}_z$ is the unit vector; the f_n are weighting functions whose logarithmic derivatives are denoted above by $g_n(z)$. In order to be able to represent the equations of motion of the medium in the form (1.1), it is sufficient to require that its parameters λ, μ, ρ satisfy the system of two nonlinear differential equations

$$(\mu g_1)' - \mu g_1^2 = ((\lambda + 2\mu)\,g_2)' - (\lambda + 2\mu)\,g_2^2 = E\rho, \tag{1.2}$$

where E is a constant. In the integration of (1.2) one function should be regarded as known, and for the other two we specify the function values and derivative values on the surface of the half space, z = 0. We determine the fifth constant from an additional condition. If we take this to be equal to zero and take for the fundamental parameters of the medium the Poisson coefficient ν, the shear modulus μ, and the square of the velocity of the shear waves w, then the system (1.2) can be integrated completely, as a result of which we obtain [2]

$$\mu(z) = \mu(0)(1 + Az)^{-1}\exp\left(B\int_0^z(1 + B(z - J(z)))^{-1}dz\right), \tag{1.3}$$

$$w(z) = w(0)(1 + Az(1 + B(z - J(z))), \; J(z) = \int_0^z \frac{1 - 2\nu(z)}{1 - \nu(z)}dz,$$

$$B = (a + b)(1 - \nu(0)), \; A = B - b, \; a = w'(0)w^{-1}(0), \; b = \mu'(0)\mu^{-1}(0).$$

The functions μ, w will be bounded and positive if a, b satisfy the conditions $a + b \geq 0$, $b \leq a(1 - \nu(0))\nu^{-1}(0)$. In the case of a constant Poisson coefficient, the system of equations (1.2) reduces to a single second-order nonlinear equation which depends on two arbitrary constants. The particular solutions of this equation are not covered by the functions in (1.3). Examples of these solutions are given in Table 1.

Since ν varies from zero to 0.5, it can be regarded without loss of generality as a slowly varying function. The other parameters of the medium will also be slowly varying functions if a and b, defined in (1.3), are considered to be small quantities. In that case the functions $H_n(z)$, defined in (1.1), will be proportional to the squares of small parameters, and since the $H_n(z)$ are expressed in terms of the logarithmic derivatives of the functions μ, ρ, they will decrease with increasing z, tending to a constant, in particular to zero. Because of this, we shall disregard the $H_n(z)$ terms.

Suppose that on a half space with the properties described above there is placed a plate whose deflection W is described by the equation

$$D\Delta\Delta W + \rho_1 h\partial^2 W/\partial t^2 = q - p,$$

where h, ρ_1 are the thickness and density of the plate; D, cylindrical rigidity; q, intensity of the external load; p, reaction of the half space. Suppose that the plate is acted upon by a vibrating load $qe^{i\omega t}$ applied to a body of mass occupying a rectangular region with sides l_2, and moving at velocity c in the positive direction of the axis OX. At the contact boundary between the plate and the half space it is assumes that there are no tangential stresses and that the conditions

$$W(x, y, t) = u_z(x, y, 0, t), \; p(x, y, t) = \sigma_z(x, y, 0, t); \tag{1.4}$$

TABLE 1

№	1		2
ν	$\nu \neq 0{,}25$	$\nu = 0{,}25$	$\nu \neq 0$
$h_n(z)$	$h_1 = 1 + \frac{az}{4\nu - 1}$	$h_2 = 1 + \frac{bz}{2}$	$h_3 = 1 + az$
$\frac{\mu(z)}{\mu(0)}$	$h_1^{3-4\nu}$	h_2^2	$h_3^{(1-\nu)/\nu}$
$\frac{\rho(z)}{\rho(0)}$	$h_1^{4(1-2\nu)}$	h_2^2	$h_3^{(1-2\nu)/\nu}$

are satisfied, where u_z, σ_z are the normal displacement and stress, respectively. It is required to find the deflection of the plate and the stressed state of the half space. The problem is solved in a moving system of coordinates $\xi = x - ct$, $y = y$. After applying a Fourier transform to the equations (1.1) with respect to ξ and y, we obtain ordinary differential equations with slowly varying coefficients. If the $v_n^2(z)$ increase at infinity more slowly than a second-degree polynomial, then the solutions of these equations can be represented by an absolutely convergent series at any finite point of the semiaxis $z \geq 0$. If, on the other hand, the rate of growth of the $v_n^2(z)$ is quadratic or higher, then the solution is represented by an asymptotic series [3]. In either case we shall use the first terms of these series, or, in other words, the WKB solution [4]. Taking account of this, we can represent the solution of the system (1.1) in the form

$$\Psi_n(\xi, y, z, t) = e^{i\omega t} \iint_{-\infty}^{\infty} G_n(\alpha, \beta) L_n(z) e^{i(\alpha\xi + \beta y)} d\alpha d\beta, \tag{1.5}$$

$$L_n(z) = \eta_n^{1/2}(\alpha, \beta, 0)\, \eta_n^{-1/2}(\alpha, \beta, z) \exp\left(-\int_0^z \eta_n(\alpha, \beta, z)\, dz\right),$$

$$\eta_n(\alpha, \beta, z) = (\alpha^2 + \beta^2 - v_n^{-2}(z)(\omega - \alpha c)^2)^{1/2} \quad (n = 1, 2),$$

where α, β are the parameters of the Fourier transform; the G_n are found from the boundary conditions; ω is the frequency of the vibrations. It should be noted that when $a = b = 0$ and ν is constant, the equations (1.1) become the equations of motion of a homogeneous medium and the equations (1.5) become their exact solutions. The deflection of the plate can also be represented as a double Fourier integral:

$$W(\xi, y, t) = e^{i\omega t} \iint_{-\infty}^{\infty} \psi(\alpha, \beta) e^{i(\alpha\xi + \beta y)} d\alpha d\beta. \tag{1.6}$$

Satisfying the boundary conditions (1.4) and taking account of the absence of tangential stresses, we find

$$\psi(\alpha, \beta) = \frac{q(\alpha, \beta)(1 + m\omega^2 V)}{\pi^2 D \Omega(\alpha, \beta)}; \tag{1.7}$$

$$G_n = \frac{\psi(\alpha, \beta)}{\varkappa(\alpha, \beta)} \psi_{21}^{n-1}(\alpha, \beta)\, \psi_{22}^{2-n}(\alpha, \beta) \quad (n = 1, 2),\ G_3 = 0, \tag{1.8}$$

where

$$\Omega(\alpha, \beta) = (\alpha^2 + \beta^2)^2 - \frac{\rho_1 h}{D}(\omega - c\alpha)^2 + \frac{\mu(0) F(\alpha, \beta)}{D\varkappa(\alpha, \beta)};$$

$$V = I(1 + \omega^2 m I)^{-1};\ I = \frac{1}{\pi^2} \iint_{-\infty}^{\infty} q(\alpha, \beta)\, \Omega^{-1}(a, \beta)\, d\alpha d\beta;$$

$$\varkappa(\alpha, \beta) = k_2^2 \eta_1(\alpha, \beta) + ((\alpha^2 + \beta^2) g_2 - \eta_1(\alpha, \beta)\, \eta_2(\alpha, \beta)\, g_1) -$$

$$-(\alpha^2 + \beta^2 - 0.5 k_2^2)(g_1 - g_2);\ F(\alpha, \beta) \equiv \psi_{11}(\alpha, \beta)\, \psi_{22}(\alpha, \beta) - \psi_{21}(\alpha, \beta)\, \psi_{12}(\alpha, \beta);\ k_2 = \omega v_2^{-1}(0);$$

$$\psi_{nn}(\alpha, \beta) = 2(\alpha^2 + \beta^2) - k_2^2 + \gamma^{2(2-n)} g_n \eta_n(\alpha, \beta);$$

$$\psi_{12}(\alpha, \beta) = -(\alpha^2 + \beta^2)(2\eta_2(\alpha, \beta) + \gamma^2 g_2 + g_1 - g_2);$$

$$\psi_{21}(\alpha, \beta) = -(2\eta_1(\alpha, \beta) + g_2);\ \gamma = \gamma(0);$$

$$g_1 = g_1(0) = 2[b\nu(0) - a(1 - \nu(0))];\ g_2 = g_2(0) = (1 - 2\nu(0))(a + b).$$

We denote by $q(\alpha, \beta)$ the Fourier transform of the pressure intensity $q(\xi, y)$. If the pressure is uniformly distributed in the rectangle, then

$$g(\alpha, \beta) = 4 \sin (\alpha l_1/2) \sin (\beta l_2/2)(\alpha\beta)^{-1}.$$

The expression (1.7) is obtained with due regard for the force of inertia of the vibrating body but disregarding the force of gravity on it. In the expressions $\psi_{nm}(\alpha, \beta)$ we have disregarded terms of second order in the parameters a and b. Once we have $\psi(\alpha, \beta)$, we find from (1.6) the deflection of the plate and from (1.5) the potentials Ψ_n, with the aid of which we can find the displacements and stresses in the half space. Thus, for the displacements we obtain

$$u_n(\xi, y, z, t) = \sqrt{\frac{\rho(0)}{\rho(z)}} \iint_{-\infty}^{\infty} (i\alpha)^{2-n} (i\beta)^{n-1} \times \tag{1.9}$$

$$\times [\psi_{22}(\alpha, \beta) L_1(z) - \psi_{21}(\alpha, \beta)(\eta_2(\alpha, \beta, z) + \varepsilon(z)) L_2(z)] \frac{\psi(\alpha, \beta)}{\varkappa(\alpha, \beta)} e^{i(\alpha\xi + \beta y - \omega t)} d\alpha d\beta,$$

$$u_z(\xi, y, z, t) = \sqrt{\frac{\rho(0)}{\rho(z)}} \iint_{-\infty}^{\infty} [\psi_{22}(\alpha, \beta)(\eta_1(\alpha, \beta, z) - \varepsilon(z)) L_1(z) - \psi_{21}(\alpha, \beta) L_2(z)] \frac{\psi(\alpha, \beta)}{\varkappa(\alpha, \beta)} e^{i(\alpha\xi + \beta y - \omega t)} d\alpha d\beta,$$

$$u_1 = u_x, u_2 = u_y, \varepsilon(z) = \frac{1}{2}(g_1(z) - g_2(z)).$$

2. We consider some special cases of the expressions obtained above. Suppose that there is applied to the half space a vibrating load distributed in the strip $|x| < l$ with intensity $q = q(x)$. The displacements in the half space are obtained from (1.9) by taking $D = m = c = \beta = h = 0$ and leaving one integral with respect to α:

$$u_n(x, z, t) = \sqrt{\frac{\rho(0)}{\rho(z)}} e^{i\omega t} \int_{-\infty}^{\infty} (i\alpha)^{n-1} K_n(\alpha, z) \frac{q(\alpha)}{F(\alpha)} e^{i\alpha x} d\alpha, \tag{2.1}$$

$$K_n(\alpha, z) = (-1)^n \psi_{22}(\alpha) Q_n^{2-n}(z) L_1(z) + (-1)^{3-n} \alpha^{2(2-n)} \psi_{21}(\alpha) Q_n^{n-1}(z) L_2(z),$$

$$Q_n(z) = \eta_n(\alpha, z) + (-1)^n \varepsilon(z), u_1 = u_z, u_2 = u_x;$$

$$F(\alpha) = F_0(\alpha) - k_2^2(\gamma^2 g_2 \eta_1(\alpha) + g_1 \eta_2(\alpha)) - 2\varepsilon(0)\alpha^2(\eta_1(\alpha) - \tag{2.2}$$

$$- \eta_2(\alpha)) - \alpha^2(\gamma^2 g_1 - (\gamma^2 - 1)\varepsilon(0)) g_2 + \gamma^2 \eta_1(\alpha) \eta_2(\alpha) g_1 g_2,$$

$$F_0(\alpha) = (2\alpha^2 - k_2^2)^2 - 4\alpha^2 \eta_1(\alpha) \eta_2(\alpha), \eta_n(\alpha) = \eta_n(\alpha, 0).$$

In the above formulas, similar quantities have been given the symbols previously used. For example, $q(\alpha)$ is the Fourier transform of the intensity $q(x)$, and $\psi_{nm}(\alpha)$, $\eta_n(\alpha, z)$, $L_n(z)$ can be found from (1.5) if the indicated parameters are taken equal to zero in those equations. The Rayleigh function $F(\alpha)$ consists of the corresponding function of a homogeneous medium, $F_0(\alpha)$, and the terms resulting from the inhomogeneity of the medium with respect to z. If the condition $4\mu'(z)\rho(z)(1 - 2\nu(z)) = \mu(z)\rho'(z)(3 - 4\nu(z))$ is satisfied, then $\varepsilon(0) = 0$ and (2.2) is simplified. In particular, this is true for the medium described in the first column of Table 1 (medium 1), and its Rayleigh equation is discussed in [5, 6]. The root of the Rayleigh equation $F(\alpha) = 0$, which ensures the existence of the surface waves, can be found by the method of successive approximations if we take the root of the equation $F_0(\alpha) = 0$ as the zeroth approximation. It is convenient to write $\alpha = k_2\zeta$ and divide the equation $F(\alpha) = 0$ by the common factor k_2^4. Then the root of the resulting equation, ζ_R, which ensures the existence of the surface waves, will be a single-valued and continuous function of the dimensionless parameters ak_2^{-1} and bk_2^{-1} in a neighborhood of the Rayleigh root of the equation $F_0(\alpha) = 0$. This is a consequence of the theorem on the existence of an explicit function [7], and the fact that this is satisfied is easy to verify. Thus, for medium 1, when $\nu = 0.25$ and $ak_2^{-1} = 0.2$ we have $\zeta_R = 1.0365217$. The corresponding value of the root for a homogeneous medium is 1.08885. Since the rigidity of the medium will increase as a increases, we should expect an increase in the velocity of the Rayleigh wave, v_R, and this is indeed the case, since $v_R = v_2(0)\zeta_R^{-1}$. We should also point out another effect resulting from the inhomogeneity of the medium. The real roots of the equation (2.2), one of which corresponds to the surface waves, exist for $\zeta \leq 1$. Taking $\zeta = 1$, we obtain from (2.2) a value of the frequency ω below which there is no excitation of Rayleigh waves in the medium without absorption of energy:

$$\omega \geqslant v_2(0)\left[\left(\frac{\gamma^2 - 1}{\gamma^2}\right)^{\frac{1}{2}} (\gamma^2 g_2 + 2\varepsilon(0)) + \frac{1}{2}\left(\frac{\gamma^2 - 1}{\gamma^2}(\gamma^2 g_2 + 2\varepsilon(0))^2 + 4(\gamma^2 g_1 - (\gamma^2 - 1)\varepsilon(0)) g_2\right)^{\frac{1}{2}}\right]. \tag{2.3}$$

In particular, for medium 1 we have $g_1 = g_2 = 2a(1 - 2\nu)$, $\varepsilon(0) = 0$, and (2.3) coincides with the result obtained in [5] for $\nu = 0.25$.

The calculation of integrals of the type (2.1) by contour integration in the complex α plane has been discussed in detail in [8]. Here we shall only point out that $F(\alpha)$ has a different forms depending on the interval in which α is situated. In particular, in the intervals $(0, k_1)$, (k_1, k_2) we have

$$F_m(\alpha) = A_m(\alpha) + iB_m(\alpha);$$
$$\text{for } m = 1 \quad \alpha \in (0, k_1), \eta_n(\alpha) = i\sqrt{k_n^2 - \alpha^2}, k_n = \omega/v_n(0);$$
$$\text{for } m = 2 \quad \alpha \in (k_1, k_2), \eta_1(\alpha) = \sqrt{\alpha^2 - k_1^2}, \eta_2(\alpha) = i\sqrt{\alpha^2 - k_2^2},$$

where i is the imaginary unit; A_m, B_m are, respectively, the real and imaginary parts of $F(\alpha)$ in the intervals mentioned, which are found from (2.2).

In working out methods for insulating machinery from vibrations in some geophysical experiments and other cases, it is important to know the energy characteristics of the elastic half space; the total power of the elastic waves caused by the surface vibration source, the power expended on the excitation of the longitudinal, transverse, and Rayleigh waves, taken individually, and the directional diagram of the energy radiated. We shall now derive the functions which contain the information concerning the above-mentioned questions, and we generalize the analogous results obtained for homogeneous media [8]. The quantity N, the average value taken over the period for the power transmitted to the half space in the case of loading with a normal harmonic force having amplitude $2\mu(0)q(x)$ in the interval $|x| < l$, can be calculated from the formula

$$N = -\mu(0)\,\omega \int_{-l}^{l} q(x)\,\mathrm{Im}\,u_z(x, 0)\,dx,$$

where the symbol Im indicates taking the imaginary part of the normal displacement u_z. The latter is found from (2.1), where we must set n = 1, z = 0. For $K_1(\alpha, 0)$ we obtain

$$K_1(\alpha, 0) = k_2^2\eta_1(\alpha) + (\alpha^2 g_2 - \eta_1(\alpha)\,\eta_2(\alpha)\,g_1) - (2\alpha^2 - k_2^2)\,\varepsilon(0), \tag{2.4}$$

$K_1(\alpha, 0)$ also changes its form, depending on the interval of integration. For $\alpha \in (0, k_1)$ and $\alpha \in (k_1, k_2)$ the radicals $\eta_n(\alpha)$ can be written analogously to the preceding case. Therefore in the above mentioned intervals we have

$$K_{1m}(\alpha, 0) = C_m(\alpha) + iD_m(\alpha),$$

where C_m and D_m are the real and imaginary parts of $K_1(\alpha, 0)$ in the respective intervals, which are found from (2.4). Taking account of the foregoing, we represent the desired power N in the form

$$N = \frac{\omega}{16\mu\pi}(4\mu(0)\,l)^2\left\{-2\pi\left(q^2(\alpha)\frac{K_1(\alpha)}{\frac{\partial F(\alpha)}{\partial\alpha}}\right)_{\alpha=\alpha_R} + 2\int_0^{k_1} q^2(\alpha)\frac{C_1(\alpha)A_1(\alpha) + D_1(\alpha)B_1(\alpha)}{A_1^2(\alpha) + B_1^2(\alpha)}\,d\alpha + \right. \tag{2.5}$$
$$\left. + 2\int_{k_1}^{k_2} q^2(\alpha)\frac{C_2(\alpha)B_2(\alpha) + D_2(\alpha)A_2(\alpha)}{A_2^2(\alpha) + B_2^2(\alpha)}\,d\alpha\right\},$$

where α_R is the root of the Rayleigh equation (2.2). In the limiting case, when $q(x) = q_0$, $4\mu(0)q_0 l \to P$, $l \to 0$, formula (2.5) yields the average power with which the concentrated force $Pe^{i\omega t}$ normal to the surface does work to excite waves in the half space, and it generalizes the well-known result obtained by Lamb [9]. As can be seen from (2.5), the power expended on the excitation of the surface waves is distinguished from the power N_1 expended on the excitation of the longitudinal and transverse waves. In some cases the power N_1 must also be separated into components, and this cannot be done by the above method. We calculate N_1 in another way – as the power flux passing through a cylindrical surface of large radius. The average value over the period, N_1, for the power passing through a semicylindrical surface of arbitrary radius R is determined from the formula

$$N_1 = -\frac{i\omega}{2}\int_0^{\frac{\pi}{2}} (\sigma_R\bar{u}_R - \bar{\sigma}_R u_R + \sigma_{R\theta}\bar{u}_\theta - \bar{\sigma}_{R\theta}u_\theta)\,R\,d\theta,$$

where σ_R, $\sigma_{R\theta}$, u_R, u_θ are the stresses and displacements, written in spherical coordinates, and the bar above a letter indicates passage to complex-conjugate quantities. For large R we can use an asymptotic representation for the displacements and stresses. Calculating the integrals (2.1) by the stationary-phase method [10], by the formulas

$$u_R = u_z\cos\theta + u_x\sin\theta,\quad u_\theta = u_x\cos\theta - u_z\sin\theta$$

we pass to the displacements in spherical coordinates, and the asymptotic behavior of the stresses is found in accordance with the formulas

$$\sigma_R = i\mu(z)k_1u_R + O(R^{-3/2}),\ \sigma_{R\theta} = \mu(z)k_2u_\theta + O(R^{-3/2}),$$

where the symbol O denotes the order of smallness of the terms that follow, which we shall disregard hereafter. For an estimate of the integrals (2.1), we must find a stationary point of the phase function

$$\int_0^z \eta_n(\alpha, z)\,dz - i\alpha x, \tag{2.6}$$

which is difficult to do, owing to the presence of the integral. Taking the parameters a and b to be small, we estimate the indicated integral by the freezing method, fixing the variable z at the level of the upper limit of integration. It is readily seen that for an arbitrarily small number ε_1 we can find values of the parameters a and b such that the inequality

$$\left|\exp\left(-\int_0^z \eta_n(\alpha, z)\,dz\right) - \exp(-z\eta_n(\alpha, z))\right| < \varepsilon_1$$

will be satisfied on the entire half axis $z \geq 0$, Carrying out the approximate integration in (2.6) and changing to the variables $R = \sqrt{x^2 + z^2}$, $\theta = \operatorname{arctg}(x/z)$, we obtain the values for the stationary points α_{0n}:

$$\alpha_{0n} = k_2\zeta_n,\ \zeta_n = r_n\gamma^{n-2}\ (n = 1, 2),$$
$$r_n = \varphi_n \sin\theta,\ \varphi_n = v_n(0)\,v_n^{-1}(R\cos\theta),$$

where the v_n are the velocities of the longitudinal and transverse waves. After performing the above operations, we obtain the asymptotic representations for the displacements in spherical coordinates:

$$u_n = u_n^{(1)} + O\left(\frac{\varepsilon(R\cos\theta)}{\sqrt{R}}\right) + O\left(\frac{1}{R^{3/2}}\right), \tag{2.7}$$
$$u_n^{(1)} = \sqrt{\frac{2}{\pi R}}\,k_n^{3,5}\frac{q(k_nr_n)}{F(k_nr_n)}\begin{pmatrix}\psi_{22}(k_1r_1)\\ \psi_{21}(k_2r_2)\end{pmatrix}\left[\varphi_n\begin{pmatrix}\cos\theta\\ \sin 2\theta\end{pmatrix} - \varepsilon(R\cos\theta)(-1)^n\right]\exp\left(iRk_n\varphi_n - \frac{i(2n+1)}{4}\pi\right),\ u_1^{(1)} = u_R,\ u_2^{(1)} = u_\theta,$$

where for n = 1 we take the upper function in the parentheses, and for n = 2 the lower; the symbol $O(\varepsilon/\sqrt{R})$ denotes terms which are proportional to $\varepsilon(R\cos\theta)$ and decrease as $1/\sqrt{R}$ but constitute an alternative wave in comparison with the first term. Consequently, in the case of an inhomogeneous medium the radial displacement is formed not only by the longitudinal wave but also by the transverse wave, and this is also true of the formation of the circular displacement. If $\varepsilon(R\cos\theta)$ is of the order of $1/R$, then the second terms, like the subsequent ones, may be disregarded. In this case the power N_1 can be subdivided into its components. We shall assume that this is the case we are dealing with, and from this point on we shall consider only the first terms in (2.7). We find specific expressions for the functions $F(k_nr_n)$ on the assumption that the φ_n decrease with increasing R:

$$F(k_1r_1) = k_1^4(A_3 + iB_3),$$
$$A_3 = (2r_1^2 - \gamma^2)^2 + 4r_1^2e_1e_2 - k_2^{-2}[r_1^2(\gamma^2g_1 - (\gamma^2 - 1)\varepsilon(0)g_2 + \gamma^2e_1e_2g_1g_2,\ B_3 =$$
$$= k_1^{-1}[\gamma^2(\gamma^2g_2e_1 + g_1e_2) + 2\varepsilon(0)(e_1 - e_2)],\ e_1 = (1 - r_1^2)^{1/2},\ e_2 = (\gamma^2 - r_1^2)^{1/2}.$$

As the angle θ varies from zero to $\pi/2$, the radicals e_n remain real, and therefore the form of the real and imaginary parts is retained; the function $F(k_2r_2)$ behaves differently:

$$F_n(k_2r_2) = k_2^4(A_{3+n} + iB_{3+n}) \qquad (n = 1, 2),$$
$$0 \leqslant \theta \leqslant \theta_0 = \arcsin(1/\gamma\varphi_2)(n = 1),\ \theta_0 \leqslant \theta \leqslant \pi/2(n = 2).$$

Here A_4, B_4 are obtained from A_3, B_3, if in the latter we replace r_1 with r_2, e_1 with $(1 - r_2^2)^{1/2}$ and e_2 with $(\gamma^{-2} - r_2^2)^{1/2}$. If in the function $F_1(k_2r_2)$ just obtained we replace $i(\gamma^{-2} - r_2^2)^{1/2}$ with $(r^2 - \gamma^{-2})^{1/2}$, its real part will yield A_5 and the coefficient of its imaginary part will yield B_5. Using the notation introduced earlier, we obtain

$$N_1 = N_p + N_s, \tag{2.8}$$
$$N_p = \frac{\omega\gamma^2}{8\mu(0)\pi}(4\mu(0)\,l)^2\int_0^{\pi/2} q^2(k_1r_1l)\,J_p(\theta)\,d\theta,$$
$$J_p(\theta) = \frac{[(2r_1^2 - \gamma^2)^2 + k_1^{-2}g_2^2(\gamma^2 - r_1^2)](\varphi_1^2\cos^2\theta + k_1^{-2}\varepsilon^2)}{A_3^2 + B_3^2};$$

$$N_s = \frac{\omega}{8\mu(0)\pi}(4\mu(0)\,l)^2\left\{\int_0^{\theta_0} q^2(k_2r_2l)\,J_{1s}(\theta)\,d\theta + \int_{\theta_0}^{\frac{\pi}{2}} q^2(k_2r_2l)\,J_{2s}(\theta)\,d\theta\right\}, \tag{2.9}$$

$$J_{1s}(\theta) = \left(\gamma^{-2} - r_2^2 + \frac{g_2^2}{4k_2^2}\right)\left(\varphi_2^2\sin^2 2\theta + \frac{\varepsilon^2(R\cos\theta)}{k_2^2}\right)(A_4^2 + B_4^2)^{-1},$$

$$J_{2s}(\theta) = \left(\sqrt{r_2^2 - \gamma^{-2}} + \frac{g_2}{2k_2}\right)^2\left(\varphi_2^2\sin^2 2\theta + \frac{\varepsilon^2(R\cos\theta)}{k_2^2}\right)(A_5^2 + B_5^2)^{-1}.$$

We note two effects produced by the inhomogeneity of the medium, assuming for the sake of definiteness that $q(x) = q_0$. Then $q(\alpha) = 2q_0\sin(\alpha l)\alpha^{-1}$. From the equation $\sin(\alpha_R l) = 0$ we find the values l_n such that the Rayleigh wave is not excited: $l_n = n\pi(k_2\zeta_R)^{-1}$ $(n = 1, 2, 3\ldots)$. Since ζ_R depends on the above-mentioned inhomogeneity parameters of the medium, ak_2^{-1}, bk_2^{-1}, for the same frequency ω there will be different values of l_n for homogeneous and inhomogeneous media. Thus, if we take medium 1 and use the Rayleigh roots mentioned above, the values of l_n will differ by 5%.

From the equation $\sin(lk_nr_n) = 0$ we find the direction along which the longitudinal or transverse wave is not propagated. Let Δ_n denote the angle of rotation of this direction, resulting from the inhomogeneity of the medium. Then

$$\Delta_n = \arcsin\left(\frac{k\pi l^{-1}}{k_n\varphi_n\sin\theta}\right) - \arcsin\left(\frac{k\pi l^{-1}}{k_n\sin\theta}\right),\quad k = 1, 2, 3\ldots,\quad n = 1, 2,\ \theta \neq 0.$$

Formulas (2.5), (2.8), and (2.9) generalize the corresponding results for homogeneous media and tend to those results as a and b tend to zero.

3. We consider a second special case of loading; in order to avoid undue complications, we shall confine our attention to medium 1. Suppose that to the surface of the half space there is applied a normal pressure of constant intensity along a line parallel to the axis OX, moving with velocity c. The displacements in the half space can be obtained from (1.9) by taking the corresponding parameters equal to zero and leaving one integral with respect to α. Calculating this on the assumption that $c < v_2(0)$, $1 - c^2v_n^{-2}(z) > 0$, we obtain

$$u_n(\xi, z, t) = \sqrt{\frac{\rho(0)}{\rho(z)}}\sum_{k=1}^{2}\frac{d_{nk}}{F_0(c)}\left\{m_nM_{nk} + \frac{1}{\alpha_1 - \alpha_2}\sum_{s=1}^{2}(-1)^s(m_n\alpha_s + a_1\delta_2^{2-n})\Big[M_{nk} - 0.5e^{-\alpha_s\chi_n(z)}\Big[(\Gamma(0, \alpha_s\tau_{1n}) + \Gamma(0, \alpha_s\tau_{2n}))\times\right.$$

$$\left.\times\begin{pmatrix}\cos(\alpha_s\xi)\\ -\sin(\alpha_s\xi)\end{pmatrix} + i(\Gamma(0, \alpha_s\tau_{1n}) - \Gamma(0, \alpha_s\tau_{2n}))\begin{pmatrix}\sin(\alpha_s\xi)\\ \cos(\alpha_s\xi)\end{pmatrix}\Big]\Big]\right\},$$

$$\tau_{1n} = \chi_n(z) + i\xi,\ \tau_{2n} = \chi_n(z) - i\xi,\ \chi_n(z) = \int_0^z \delta_n(c, z)\,dz,$$

$$M_{n1} = \ln(\xi^2 - \chi_n^2(z))^{1/2},\ M_{n2} = \pi - 2\,\mathrm{arctg}\,\frac{\chi_n(z)}{\xi},$$

$$\alpha_s = \frac{a}{F_0(c)}\left[k_c^2(\gamma^2\delta_1 + \delta_2) \pm (k_c^4(\gamma^2\delta_1 + \delta_2)^2 + 4F_0(c)(1 - \delta_1\delta_2))^{1/2}\right],$$

$$F_0(c) = (2 - k_c^2)^2 - 4\delta_1\delta_2,\ \delta_n(z) = \left(1 - \frac{c^2}{v_n^2(z)}\right)^{1/2},$$

$$d_{n1} = \delta_1^{2-n}(z)\,\delta_n^{1/2}(0)\,\delta_n^{-1/2}(z),\quad d_{n2} = \delta_2^{n-1}(z)\,\delta_n^{1/2}(0)\,\delta_n^{-1/2}(z),$$

$$m_1 = 2 - k_c^2,\quad m_2 = 2\delta_1(0),\quad k_c = cv_2^{-1}(0),$$

$$a_1 = 2a(1 - 2\nu(0)),\ \delta_n = \delta_n(0),\ u_1 = u_z,\ u_2 = u_x,$$

where $\Gamma(0, \tau)$ is the incomplete gamma function. For n = 1 we must take the upper trigonometric function, and for n = 2 the lower. The term $M_{nk}m_n$ in the case of a homogeneous medium yields the well-known result of [11]. The remaining terms are due to the inhomogeneity of the medium. The normal displacement u_z, as in the case of a homogeneous medium, has a logarithmic singularity, but the discontinuity surface is deformed. The additional terms may be interpreted as deformation waves caused by the inhomogeneity of the medium and having a period of $2\pi\alpha_s^{-1}$. As $a \to 0$, the amplitudes of these waves tend to zero and the period of the vibrations tends to infinity.

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REALIZATION OF NONAXISYMMETRICAL MOMENT-FREE STATE IN SHELLS OF REVOLUTION

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A series of formulations of problems involving realization of a moment-free stressed state in elastic reinforced shells with arbitrary shape of the center surface is given in [1]. This paper is concerned with solving three of the problems proposed in [1] for the case when the center surface of the shell is a surface of revolution with nonzero Gaussian curvature. The problem of the possibility of realizing a moment-free state in arbitrary reinforced shells with zero curvature was examined in [2] and the particular case of axial symmetry was examined in [3].

1. We shall examine a shell of revolution with a quasiuniform layered structure over the thickness. We shall choose a system of coordinates fixed to the lines of principle curvature of the surface of the shell. If the shell functions in a moment-free stress state, then the following relations must be satisfied [1]:

equations of equilibrium

$$\begin{gathered}\partial(rT_1)/\partial\varphi - T_2R_1\cos\varphi + R_1\partial T_{12}/\partial\theta = -rR_1p_1,\\ R_1\partial T_2/\partial\theta + \partial(rT_{12})/\partial\varphi + T_{12}R_1\cos\varphi = -rR_1p_2,\\ T_1R_2 + T_2R_1 = R_1R_2p_3;\end{gathered} \quad (1.1)$$

elasticity relations

$$\begin{gathered}T_1 = h(a_{11}\varepsilon_1 + a_{12}\varepsilon_2 + a_{13}\varepsilon_{12}),\ T_2 = h(a_{12}\varepsilon_1 + a_{22}\varepsilon_2 + a_{23}\varepsilon_{12}),\\ T_{12} = T_{21} = h(a_{13}\varepsilon_1 + a_{23}\varepsilon_2 + a_{33}\varepsilon_{12});\end{gathered} \quad (1.2)$$

geometric equations

$$\begin{gathered}\varepsilon_1 = \frac{1}{R_1}\frac{\partial u}{\partial\varphi} + \frac{w}{R_1},\quad \varepsilon_2 = \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{\cos\varphi}{r}u + \frac{w}{R_2},\\ \varepsilon_{12} = \frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{r}{R_1}\frac{\partial}{\partial\varphi}\left(\frac{v}{r}\right);\end{gathered} \quad (1.3)$$

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